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A LOGICAL APPROACH TO DECIDABILITY OF HIERARCHIES OF REGULAR STAR–FREE LANGUAGES

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We propose a new, logical, approach to the decidability problem for the Straubing and Brzozowski hierarchies based on the preservation theorems from model theory, on a theorem of Higman, and on the Rabin tree theorem. In this way, we get purely logical, short proofs for some known facts on decidability, which may be of methodological interest.

Our approach is also applicable to some other similar situations, say to "words" over dense orderings which is relevant to the continuous time and hybrid systems.

Keywords: star–free regular languages, hierarchies, definability, decidability.

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ЛОГИЧЕСКИЙ ПОДХОД К РАЗРЕШИМОСТИ ИЕРАРХИЙ РЕГУЛЯРНЫХ БЕЗЗВЕЗДНЫХ ЯЗЫКОВ

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Представляем новый, логический подход к проблеме разрешимости иерархий Страубинга и Бржозовского, основанный на теореме сохранения из теории моделей, на теореме Хигмана и на теореме о дереве Рабина. Таким образом мы получаем чисто логические краткие доказательства некоторых известных фактов о разрешимости, которые могут представлять методологический интерес. Наш подход также применим к некоторым другим похожим ситуациям, скажем, к “словам” над плотными порядками, относящимся к непрерывному времени и гибридным системам.

Ключевые слова: беззвездные регулярные языки, иерархии, определимость, разрешимость.

1. INTRODUCTION

In the automata theory, several natural hierarchies of regular languages are studied. Among the most popular are hierarchies of Brzozowski and Straubing [Pin86], both exhausting the regular star–free languages. A natural question about these hierarchies is formulated as follows: given a level of a hierarchy and a finite automaton, one has to decide effectively whether or not the language of the automaton is in the given level. Till now, this question is solved positively only for lower levels. For higher levels, the problem is still open and seems to be hard (see, e.g., [Pin86, Pin94] for more information and references).

In the literature one could identify at least two approaches to the decidability problem, which may be called algebraic and automata–theoretic. The first approach exploits the well-known relationship between regular languages and semigroups, the second one tries to find a property of a finite automaton (usually in terms of so-called forbidden patterns) equivalent to the property that the language recognized by the automaton is in the given level.

In this paper, we propose another, logical approach to the problem. From [Th82, PP86], it follows that the problem might be formulated similarly to some traditional decidability problems of logic. Our main observation is that in this situation one can apply some old facts known as preservation theorems (see, e.g., [Ro63, Ma71]), as well as a theorem of Higman [CKa91]. Observing that the corresponding conditions are interpretable in the Rabin tree theory, we get new, purely logical and short proofs of some known facts on decidability. This may be of methodological interest. Our approach is also applicable to some other similar situations and yields several new results.

The rest of our paper is organized as follows: we consider some versions of the Straubing hierarchy in Section 2, some versions of the Brzozowski hierarchy in Section 3, the role of the empty word and relationships between our versions and the "real" Straubing and Brzozowski hierarchies in Section 4, some relevant results and possible directions of future work in Section 5.

We close this introduction with reminding notation used throughout the paper. Let A be an alphabet, i.e. a finite nonempty set. Let $A^*(A^+)$ be the set of all words (resp., of all nonempty words) over A. As usual, the empty

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word is denoted by ε, the length of a word u by $|u|$, and the concatenation of words u and v by uv. The concatenation of languages X and Y is denoted by XY. For $u = u_0 \ldots u_n \in A^+$ and $i \leq j \leq n$, let $u[i,j]$ denote the segment (or factor) of u bounded by i, j (including the bounds).

2. STRAUBING-TYPE HIERARCHIES

A word $u = u_0 \ldots u_n \in A^+$ may be considered as a structure $\mathbf{u} = (\{0, \ldots, n\}; <, Q_a, \ldots)$, where $<$ has its usual meaning and $Q_a (a \in A)$ are unary predicates on $\{0, \ldots, n\}$ defined by $Q_a(i) \leftrightarrow u_i = a$. As is well-known (see, e.g., [MP71, Th82, PP86]), there is a close relationship between star-free languages and classes of models $\mathbf{u}$ of sentences of signature $\sigma = \{<, Q_a, \ldots\}$ (in this section, "sentence" means "a first-order formula of signature σ without free variables"; the only exception is the proof of Theorem 2.1 below where we also need sentences of another kind). Note that the alphabet A is fixed throughout the paper, hence we do not mention the alphabet in our notation, like σ or CLO.

Let us consider a first-order theory CLO of signature σ that is closely related to the theory of regular languages. The axioms of CLO state that $<$ is a linear ordering and any element satisfies exactly one of the predicates $Q_a (a \in A)$. Models of CLO are called colored (more precisely, A-colored) linear orderings. We use letters like $\mathbf{u}, \mathbf{v}, \ldots$ (respectively, $\mathbf{U}, \mathbf{V}, \ldots$) to denote finite (respectively, countable) models of CLO. As usual, $\mathbf{U} \subseteq \mathbf{V}$ denote that $\mathbf{U}$ is a substructure of $\mathbf{V}$. For a sentence ϕ, let M_ϕ be the set of all countable models of CLO satisfying ϕ, in a symbolic form $M_\phi = \{\mathbf{U} | \mathbf{U} \models \phi\}$. Note that any finite model of CLO is isomorphic to the structure $\mathbf{u}$ from the preceding paragraph, for a unique $u \in A^+$. The relation $\subseteq$ induces a partial ordering on A^+ that will be denoted by the same symbol.

For $n > 0$, let Σ^0_n denote the set of all sentences in the prenex normal form starting with the existential quantifier and having $n - 1$ quantifier alternations. Let S_n be the set of sentences equivalent to a Σ^0_n-sentence (modulo theory CLO). In other words, $S_n = \{\psi | \exists \phi \in \Sigma^0_n (M_\psi = M_\phi)\}$. Let $\check{S}_n$ be the dual set for S_n, i.e. the set of sentences equivalent to negations of S_n-sentences. Let $B(S_n)$ be the set of sentences equivalent to a Boolean combination of Σ^0_n-sentences. Then we have the following assertions.

Lemma 2.1. *(i) For any $n > 0$, $B(S_n) = S_{n+1} \cap \check{S}_{n+1}$.*
(ii) $\phi \in S_1$ iff $\forall \mathbf{U} \models \phi \forall \mathbf{V} \supseteq \mathbf{U} (\mathbf{V} \models \phi)$.
(iii) $\phi \in \check{S}_2$ iff the union of an arbitrary chain $\mathbf{U}_0 \subseteq \mathbf{U}_1 \subseteq \cdots$ of models

of ϕ is a model of ϕ.

(iv) $\phi \in S_2$ iff $\forall \mathbf{U} \models \phi \exists \mathbf{u} \subseteq \mathbf{U} \forall \mathbf{V}(\mathbf{u} \subseteq \mathbf{V} \subseteq \mathbf{U} \to \mathbf{V} \models \phi)$.

Proof. (i)—(iii) are well-known results of logic (see, e.g., [Ro63, Ma71, Sh67]), while (iv) easily follows from (iii). Namely, if a sentence $\phi = \exists \bar{x} \forall \bar{y} \psi(\bar{x}, \bar{y})$, where ψ is a quantifier-free formula, is true in $\mathbf{U}$, then let $\mathbf{u}$ be the substructure of $\mathbf{U}$ with the universe $\{x_1, \ldots, x_n\}$, where $\bar{x} = (x_1, \ldots, x_n)$. Then $\mathbf{u}$ clearly satisfies the condition $\forall \mathbf{V}(\mathbf{u} \subseteq \mathbf{V} \subseteq \mathbf{U} \to \mathbf{V} \models \phi)$. Conversely, assume the righthandside condition of (iv) and prove that $\phi \in S_2$. Suppose the contrary; then, by (iii), there is a chain $\mathbf{U}_0 \subseteq \mathbf{U}_1 \subseteq \cdots$ of models of $\neg\phi$ the union $\mathbf{U}$ of which satisfies ϕ. Let $\mathbf{u}$ be a finite substructure of $\mathbf{U}$ satisfying $\forall \mathbf{V}(\mathbf{u} \subseteq \mathbf{V} \subseteq \mathbf{U} \to \mathbf{V} \models \phi)$. Choosing a number i with $\mathbf{u} \subseteq \mathbf{U}_i$, one gets a contradiction (take $\mathbf{U}_i$ in place of $\mathbf{V}$).□

Let $\{D_k\}_{k \geq 0}$ be the difference hierarchy (known also as the Boolean hierarchy) over S_1. Hence, D_0 is the set of false sentences, $D_1 = S_1$, $D_2(D_3, D_4)$ is the set of sentences equivalent to sentences of the form $\phi_0 \wedge \neg\phi_1$ (respectively, $(\phi_0 \wedge \neg\phi_1) \vee \phi_2$, $(\phi_0 \wedge \neg\phi_1) \vee (\phi_2 \wedge \neg\phi_3)$) and so on, where $\phi_i \in \Sigma^0_1$ (for more information on the difference hierarchy see e.g. [Ad65, Se95]). An *alternating chain* for a sentence ϕ is by definition a sequence $\mathbf{U}_0 \subseteq \cdots \subseteq \mathbf{U}_k$ of CLO-models such that $\mathbf{U}_i \models \phi$ iff $\mathbf{U}_{i+1} \models \neg\phi$; k is called the length of such a chain. Such a chain is called a 1-alternating chain, if $\mathbf{U}_0 \models \phi$. One could consider also infinite alternating chains (with the order type ω).

The next assertions are also known in a more general form [Ad65, Se91].

Lemma 2.2. *(i) For any k, $D_k \cup \check{D}_k \subseteq D_{k+1}$.*

(ii) $\cup_k D_k = B(S_1)$.

(iii) $\phi \in D_k$ iff there is no 1-alternating chain for ϕ of length k.

We are ready to prove one of our main results on the decidability of some classes of sentences introduced above.

Theorem 2.1. *The classes of sentences $S_1, S_2, B(S_1), D_k (k \geq 0)$ are decidable.*

Proof. Let $T = \{0,1\}^*$ and let r_0, r_1 be unary functions on T defined by $r_i(u) = ui (i \leq 1)$. According to the celebrated theorem of M. Rabin [Ra69], the monadic second-order theory S2S of the structure $(T; r_0, r_1)$ is decidable. We shall use this fact in the following way: for any set $C \in \{S_1, S_2, B(S_1), D_k | k \geq 0\}$ and for any σ-sentence ϕ one can effectively construct a monadic second-order sentence $\tilde{\phi}$ of signature $\{r_0, r_1\}$ such that $\phi \in C$ iff $\tilde{\phi} \in$S2S. This is rather obviously.

We will use some well-known facts on definability (by monadic second-order formulas) in $(T; r_0, r_1)$ established in [Ra69]. First recall that the lexicographical ordering $\preceq$ on T is definable. Let $B \subseteq T$ be the set of all sequences $x101$ having no subsequence 101 except one at the end. Then B is definable and $(B; \preceq)$ has the order type of rationals. This implies that any countable linear ordering is isomorphic to an ordering of the form $(U; \preceq)$ with $U \subseteq B$. Hence, any countable model of CLO is isomorphic to a structure of the form $\mathbf{U} = (U; \preceq, Q_a, \ldots)$ with $U \subseteq B$ and $Q_a \subseteq U$ for $a \in A$ (in this proof, we call such structures *inner structures*). In the monadic logic, one can use variables for subsets of T and even quantify over them. Hence, it is possible to speak about arbitrary inner structures. We can also speak about substructures because, for any abstract models $\mathbf{U}$ and $\mathbf{V}$ of CLO, $\mathbf{U}$ is embeddable in $\mathbf{V}$ iff there are inner models $(U; \preceq, Q_a, \ldots)$ and $(V; \preceq, Q'_a, \ldots)$ isomorphic to $\mathbf{U}$ and $\mathbf{V}$, respectively, and satisfying $U \subseteq V$ and $Q_a \subseteq Q'_a (a \in A)$.

Note also that, for any fixed σ-sentence ψ, the set of all inner structures $\mathbf{U}$ satisfying ψ is definable (i.e., if ψ is $\forall x \exists y (x \leq y \wedge Q_a(y))$ then $\mathbf{U} \models \psi$ iff $\forall x \in U \exists y \in U (x \preceq y \wedge Q_a(y)))$. In particular, the set of all inner models of CLO is definable.

Now let us return to the proof of the theorem. Let, e.g., $C = S_1$ and ϕ be a given σ-sentence. Let $\tilde{\phi}$ be a sentence expressing that, for any inner model $\mathbf{U}$ of CLO satisfying ϕ, and any inner model $\mathbf{V}$ of CLO extending $\mathbf{U}$, $\mathbf{V}$ satisfies ϕ. By Lemma 2.1 and remarks above, $\phi \in S_1$ iff $\tilde{\phi} \in S2S$. This completes the proof for the set S_1. The remaining cases are treated in the same way (in the case of S_2 one shall note that the class of finite subsets of T is also definable [Ra69]). $\square$

Remark 2.1. The proof implies the known fact that the monadic second-order theory of countable models of CLO is decidable.

Theorem 2.1 demonstrates the ideas of our approach for a decision problem traditional for logic (though the results seem formally new). It turns out that, due to its abstract nature, the approach is also applicable in the context of automata theory, which we would like now to demonstrate. This application is founded on a close relationship between star–free regular languages and the first-order definability established in [MP71].

By remarks at the beginning of this section, there is a natural one-to-one correspondence between the subsets of A^+ and the classes of finite CLO-models closed under isomorphism. This induces some notions on words corresponding to notions on models introduced above; we will use some of these notions under the same names. Relate the language $L^+_\phi = \{u \in A^+ | \mathbf{u} \models \phi\}$ to any sentence ϕ. By [MP71], such languages are exactly the regular star–free languages. Let $S_n^+, B(S_n^+)$ and D_k^+ be defined as the corresponding classes above, but with L^+ in place of M; in particular, $S_n^+ = \{\psi | \exists \phi \in \Sigma_n^0 (L_\psi^+ = L_\phi^+)\}$. Then $\{B(S_n^+)\}_{n \geq 1}$ is the version of the Straubing hierarchy mentioned in the introduction.

Note that there is an evident relationship between classes $S_n^+, \dots$ and the corresponding classes without $+$, namely, $S_n \subseteq S_n^+$ and so on. But the $+$-classes contain a lot of new sentences. E.g., we have $S_1^+ \not\subseteq B(S_1)$ (the sentence saying that the ordering is dense belongs to S_1^+ but not to S_2).

Recall [CKa96, Theorem 7.2] that a *well partial ordering* is a partial ordering such that for any nonempty subset X the set of all minimal elements of X is nonempty and finite.

Lemma 2.3. *(i)* $(A^+; \subseteq)$ *is a well partial ordering.*

(ii) $\phi \in D_k^+$ *iff there is no 1-alternating chain of words for* ϕ *of length* k*.*

(iii) $\phi \in B(S_1^+)$ *iff there is no infinite alternating chain of words for* ϕ*.*

(iv) $\phi \in B(S_1^+)$ *iff* $\forall \mathbf{U} \exists \mathbf{u} \subseteq \mathbf{U} (\forall \mathbf{v} (\mathbf{u} \subseteq \mathbf{v} \subseteq \mathbf{U} \to \mathbf{v} \models \phi) \vee \forall \mathbf{v} (\mathbf{u} \subseteq \mathbf{v} \subseteq \mathbf{U} \to \mathbf{v} \models \neg\phi))$*.*

Proof. (i) is a well known result of G. Higman (see, e.g., [CKa96, Theorem 7.2]).

(ii) From left to right, the assertion follows from Lemma 2.2.(iii). Now assume that there is no 1-alternating chain of words for ϕ of length k; we have to show $\phi \in D_k^+$. For simplicity of notation, consider only the typical particular case $k = 2$; then there are no words $u_0, u_1, u_2 \in A^+$ with $u_0 \subseteq u_1 \subseteq u_2$ and $\mathbf{u}_0 \models \phi, \mathbf{u}_1 \models \neg\phi, \mathbf{u}_2 \models \phi$. Let $C_0 = \{u \in A^+ | \exists u_0 \in A^+ (u_0 \subseteq u \wedge \mathbf{u}_0 \models \phi)\}$ and $C_1 = \{u \in A^+ | \exists u_0, u_1 \in A^+ (u_0 \subseteq u_1 \subseteq u \wedge \mathbf{u}_0 \models \phi \wedge \mathbf{u}_1 \models \neg\phi)\}$. One easily checks that $L_\phi^+ = C_0 \setminus C_1$. By (i), any of C_0 and C_1 is either empty or of the form $\{v \in A^+ | v_0 \subseteq v \vee \ldots \vee v_m \subseteq v\}$ for some $m \geq 0$ and $v_0, \ldots, v_m \in A^+$. This easily implies that $C_i = L_{\phi_i}^+$ for some $\phi_i \in \Sigma_1^0 (i \leq 1)$. Then $L_\phi^+ = L_{\phi_0 \wedge \neg\phi_1}^+$. Hence, $\phi \in D_2^+$, which completes the proof.

(iii) From left to right, the assertion follows from (ii) and the equality $B(S_1^+) = \cup_k D_k^+$. It remains to show that for any $\phi \not\in B(S_1^+)$ there is an infinite alternating chain of words. By (ii), there are alternating chains of words for ϕ of arbitrary finite length.

Let ω^* be the set of all finite sequences of natural numbers, including the empty sequence ε. We construct a partial function $u : \omega^* \to A^*$ as follows. Let $u(\varepsilon) = \varepsilon$ and suppose, by induction on $|\sigma|$, that $u(\sigma)$ is already defined. If $|\sigma|$ is even then find $m \in \omega$ and words $v_0, \ldots, v_m \in A^+$ enumerating without repetitions the $\subseteq$-minimal elements in $X = \{v \in A^+ | u(\sigma) \subseteq v \wedge \mathbf{v} \models \phi\}$. Then we set $u(\sigma i) = v_i$ for $i \leq m$ and $u(\sigma i)$ is undefined for $i > m$. For $|\sigma|$ odd, the definition is similar, but we use the set $X = \{v \in A^+ | u(\sigma) \subseteq v \wedge \mathbf{v} \models \neg\phi\}$.

From (i) and (ii) easily follows that $\{\sigma \in \omega^* | u(\sigma)$ is defined$\}$ is an infinite finitely branching tree (under the relation of being an initial segment). By König's lemma, there is an infinite path through this tree. The image of this path under u provides the desired infinite alternating chain for ϕ.

(iv) Let $\phi \in B(S_1^+)$, then $L_\phi^+ = L_\psi^+$ for a Boolean combination ψ of Σ_1^0-sentences. Note that ψ and $\neg\psi$ are in S_2, and any $\mathbf{U}$ satisfies ψ or $\neg\psi$. Hence, the condition on the righthandside of (iv) follows from Lemma 2.1(iii).

Conversely, suppose that $\phi \notin B(S_1^+)$. By (i), there is an infinite alternating chain $\mathbf{u}_0 \subseteq \mathbf{u}_1 \subseteq \ldots$ for ϕ consisting of finite models of CLO. Then $\mathbf{U} = \cup_k \mathbf{u_k}$ is a countable model of CLO for which the condition on the righthandside of (iv) is false. $\square$

Repeating the proof of Theorem 2.1, one immediately gets

Theorem 2.2. *The classes $S_1^+, B(S_1^+)$ and $D_k^+ (k \geq 0)$ are decidable.*$\square$

Remark 2.2. Till now, we were unable to prove (by purely logical means) the known fact that the class S_2^+ is decidable.

Note that Lemma 2.3 and theorem 2.2 provide new, shorter proofs for several known facts from the automata theory (cf. [St85,Pin86,SW99]). E.g., decidability of $B(S_1^+)$ is equivalent (using a simple observation of Section 4 below) to the well-known result on decidability of the class of so-called piecewise testable languages.

Our method is also applicable to some other similar situations, and now we want to give a couple of examples. There are several natural modifications of the operation $\phi \mapsto L_\phi^+$, among the most popular are ω-languages $L_\phi^\omega = \{\alpha : \omega \to A | \alpha \models \phi\}$ and Z-languages (Z is the set of integers) $L_\phi^Z = \{\alpha : \omega \to A | \{\alpha \models \phi\}$, where α is the structure defined similarly to the case of finite words (one could even consider "words" over more exotic linear orderings, say, rationals or ω^2). Such operations induce the corresponding classes of sentences $S_n^\omega, B(S_n^\omega), D_k^Z$, and so on. Are such classes of sentences decidable?

Till now, we were unable to answer this question using the methods developed above. But the methods become applicable if we add finite words to the infinite ones, i.e. if we consider "languages" like $L_\phi^{\omega+} = L_\phi^\omega \cup L_\phi^+$, which are also traditional objects of automata theory, and the corresponding classes of sentences $S_n^{\omega+}, \ldots$. Let us formulate the analog of Theorem 2.2 for ω-words (similar results also hold for other kinds of infinite words).

Theorem 2.3. *The classes* $S_1^{\omega+}, B(S_1^{\omega+})$ *and* $D_k^{\omega+}(k \geq 0)$ *are decidable.*

Proofsketch. From Lemma 2.1(iii) it follows that if $\phi \in \Sigma_2^0$ then $L_\phi^{\omega+}$ is approximable (i.e., for any ω-word $\alpha \in L_\phi^{\omega+}$ there is a finite word $u \subseteq \alpha$ such that $\mathbf{v} \models \phi$ for any finite word v with $u \subseteq v \subseteq \alpha$). Repeating the proof of Lemma 1.3, one obtains analogs of assertions (ii), (iii), and (iv) for the classes $B(S_1^{\omega+}), D_k^{\omega+}(k \geq 0)$; but one have to add the condition that both $L_\phi^{\omega+}$ and $L_{\neg\phi}^{\omega+}$ are approximable to the righthandsides of these assertions.

With analog of Lemma 2.3 at hand, it is also easy to adjust the proof of Theorem 2.1 to our case. In place of the set B, we shall take now the set $B_1 = \{1^k | k < \omega\}$; it is definable and $(B_1; \preceq)$ has the order type ω. It remains to modify the notion of inner structures in such a way that their universes are subsets of B_1.$\square$

3. BRZOZOWSKI-TYPE HIERARCHIES

Here we shall consider some versions of the well-known Brzozowski hierarchy. Following [Th82] (with some minor changes), we enrich the signature σ of the preceding section to the signature $\sigma' = \sigma \cup \{\bot, \top, p, s\}$, where $\bot$ and $\top$ are constant symbols, while p and s are unary function symbols ($\bot, \top$ are assumed to denote the least and the greatest elements, while p and s are respectively the predecessor and successor functions). Let us also add to the axioms of CLO the following axioms:

$\forall x(\bot \leq x \leq \top)$,
$\forall x(p(x) \leq x \wedge \neg\exists y(p(x) < y < x)), \forall x(x \leq s(x) \wedge \neg\exists y(x < y < s(x)))$,
$\forall x > \bot(p(x) < x)$ and $\forall x < \top(x < s(x))$.

We denote the resulting theory by CLO'. For models $\mathbf{U}, \mathbf{V}$ of this theory, $\mathbf{U} \subseteq' \mathbf{V}$ means that $\mathbf{U}$ is a substructure of $\mathbf{V}$ respecting all symbols from σ'.

There is also a "relational" version of CLO' defined as follows. Let $\sigma'' = \sigma \cup \{\bot, \top, S\}$, where S is a binary predicate symbol ($\bot, \top$ are as above, while S denotes the successor predicate). Let CLO'' be obtained from CLO by adjoining the axioms

$\forall x(\bot \leq x \leq \top)$,
$\forall x, y(S(x, y) \leftrightarrow x < y \wedge \neg\exists z(x < z < y))$,
$\forall x < \top\exists y S(x, y)$ and $\forall x > \bot\exists y S(y, x)$.

Using the standard procedure of extending a theory by definable predicate and function symbols (see, e.g., [Sh67]), one can easily see that CLO' and CLO'' are essentially the same theory (e.g., every model of one theory may be, in a unique way, considered as a model of another, the natural translations respect classes of formulas S_n and analogs of other classes from Section 2, any of these classes modulo one theory is decidable if and only if it is decidable modulo the other theory, and so on). For this reason our notation will not distinguish between these theories.

It is clear that countable CLO'-models consist of all finite CLO-models and all countably infinite CLO-models of the order type $\omega + Z \cdot L + \omega^-$, where ω, ω^- and Z are, respectively, the order types of positive, negative and all integers, L is a countable (possibly empty) linear ordering, $Z \cdot L$ is the linear ordering obtained by inserting a copy of Z in place of any element of L, and $+$ is the operation of "concatenation" of linear orderings.

For the theory CLO' the analogs of Lemmas 2.1 and 2.2 hold true with some evident changes in formulation (say, the righthandside of 2.1(iv) now looks like $\forall \mathbf{U} \models \phi \exists \mathbf{u} \subseteq \mathbf{U} \forall \mathbf{V} (\mathbf{u} \subseteq \mathbf{V} \subseteq' \mathbf{U} \rightarrow \mathbf{V} \models \phi)$, where $\subseteq$ has the same meaning as in Section 2 and $\mathbf{u}$ is a finite CLO-model).

Repeating now the proof of Theorem 2.1, we immediately get the following assertion, where classes of sentences are defined just as in Section 2, but modulo theory CLO'.

Theorem 3.1. *The classes of sentences $S_1, S_2, B(S_1)$ and $D_k (k \geq 0)$ modulo theory CLO' are decidable.*

Remark 3.1. As in Section 2, the proof of Theorem 3.1 implies the decidability of the monadic second-order theory of the class of countable CLO'-models.

We see that, for the case of all countable "words", the theory CLO' is treated quite similarly to the theory CLO.

Let us now turn to finite words. The classes of sentences $S_n^+, B(S_n^+)$ and D_m^+ are defined by analogy with Section 2. Again, as in Section 2, these classes include the corresponding classes without $+$, but the converse inclusions are far from being true. E.g., the sentence $\exists x Q_a(x) \wedge \forall x > \perp (Q_a(x) \rightarrow \exists y (\perp < y < x \wedge Q_a(y)))$ belongs to S_1^+ but not to S_2.

The treatment of the $+$-classes modulo theory CLO' turns out to be more complicated, as compared with CLO. A reason is that if $\mathbf{U} \subseteq' \mathbf{V}$ and one of these CLO'-models is finite, then $\mathbf{U} = \mathbf{V}$. Hence, the analog of Lemma 2.3 is false.

In this situation, the following notion from [St85] is of some use. Let

$u = u_1 \dots u_m$ and $v = v_1 \dots v_n$ be words from A^+, where $u_i, v_j \in A$. A k-*embedding* from u to v is an increasing function $\theta : \{1, \dots, m\} \to \{1, \dots, n\}$ such that

(i) $\theta(j) = j, j = 1, \dots, min(k, m)$,
(ii) $\theta(m - j) = n - j, j = 0, \dots, min(k - 1, m - 1)$,
(iii) $u_{i+j} = v_{\theta(i)+j}, i = 1, \dots, m, j = 0, \dots, k, i + j \leq m$.

This means that u is a subword of v including the first k letters and the last k letters of v and such that any letter used to build u is followed by the same k letters in u and in v.

We write $u \leq^k v$ to denote that there is a k-embedding from u to v. For finite CLO-models $\mathbf{u}$ and $\mathbf{v}$, we write $\mathbf{u} \subseteq^k \mathbf{v}$ to denote that $\mathbf{u} \subseteq \mathbf{v}$ and the identity function is a k-embedding from u to v (u and v are words corresponding to the models as in Section 2). With some evident modifications, we may also apply the last relation to countably infinite CLO-models.

Lemma 3.1. *(i) If $u \leq^{k+1} v$ then $u \leq^k v$.*
(ii) $\leq^k$ is a partial ordering.
(iii) $\leq^0$ coincides with $\subseteq$.
(iv) If $u \leq^k v$ then $au \leq^k av$ for any $a \in A$.
(v) For all $\mathbf{u}$ and k, there is an existential σ'-sentence ϕ_u^k such that $\mathbf{u} \subseteq^k \mathbf{U}$ iff $\mathbf{U} \models \phi_u^k$.

Proof. (i)—(iv) are evident. For (v), represent u as above: $u = u_1 \dots u_m$, $u_i \in A$. If $m < 2k$, then $u \leq^k v$ if and only if $u = v$. Hence, ϕ_u^k may be the quantifier–free σ'-sentence saying that $\bot < s(\bot) < \cdots < s^{m-1}(\bot) = \top$ and the elements $\bot < s(\bot) < \cdots < s^{m-1}(\bot)$ have respectively the colors $u_1, \dots, u_m$.

Now assume $m \geq 2k$ and consider the quantifier–free sentence ψ saying that $\bot < s(\bot) < \cdots < s^{k-1}(\bot) < p^{k-1}(\top) < \dots < p(\top) < \top$, these elements have respectively the colors $u_1, u_2, \dots, u_k, u_{m-k+1}, \dots, u_{m-1}, u_m$, and the elements $s^k(\bot), \dots, s^{2k-1}(\bot)$ have respectively the colors $u_{k+1}, \dots$, u_{2k}. In the case $m = 2k$, it suffices to take $\phi_u^k = \psi$.

Finally, in the case $m > 2k$, consider the sentence θ saying that there are elements $x_{k+1} < \cdots < x_{m-k}$ such that for any $i = k + 1, \dots, m - k$ the colors of $x_i, s(x_i), \dots, s^k(x_i)$ are, respectively, $u_i, u_{i+1}, \dots, u_{i+k}$. Then we can take $\phi_u^k = \psi \wedge \theta$. $\square$

Let E^k be the set of sentences equivalent, in the theory CLO', to a finite conjunction of finite disjunctions of sentences $\phi_u^k (u \in A^+)$. Let $\{D_n^k\}_n$ be the difference hierarchy over E^k. Then we have the following analog of Lemma 2.3(i)—(iii).

Lemma 3.2. *(i)* $(A^+; \subseteq^k)$ *is a well partial ordering.*

(ii) $\phi \in E^k$ *iff* L^+_ϕ *is closed upwards under* $\leq^k$.

(iii) $\phi \in D^k_n$ *iff there is no 1-alternating* $\subseteq^k$*-chain of words for* ϕ *of length* n.

(iv) $\phi \in B(E^k)$ *iff there is no infinite alternating* $\subseteq^k$*-chain of words for* ϕ.

Proof. (i) Suppose the contrary. Then, as in the proof of [CKa96, Theorem 7.2], one can construct an infinite sequence $\{f_i\}_{i\geq 0}$ of words such that $f_i \not\leq^k f_j$ for $i < j$ and for any sequence $\{f'_i\}$ with the same property it holds that $|f_i| \leq^k |f'_i| (i \geq 0)$. From finiteness of A, it follows that there is an infinite sequence of numbers $i_0 < i_1 < \cdots$ satisfying the condition $f_{i_j} = abg_j$ for some $a, b \in A$ and $g_j \in A^* (j \geq 0)$.

Choose words $w_0, \ldots, w_{i_0-1}$ starting with a letter $c \in A\backslash\{b\}$ and pairwise incomparable under $\leq^k$. Let $\{h_i\}$ denote the sequence of words

$$w_0, \ldots, w_{i_0-1}, bg_0, bg_1, \ldots.$$

From Lemma 3.1(iv) and the choice of $w_0, \ldots, w_{i_0-1}$, it follows that $h_i \not\leq^k h_j$ for $i < j$ and $|h_{i_0}| < |f_{i_0}|$, contradicting the property of $\{f_i\}$.

(ii) Let $\phi \in E^k$. The class of subsets of A^+ closed upwards under $\leq^k$ is closed under $\cup, \cap$, hence it suffices to prove that $L^+_{\phi^k_u}$ is closed upwords. But this follows immediately from Lemma 3.1(ii),(v). Conversely, let L^+_ϕ be closed upwards. By (i), L^+_ϕ is a finite union of sets of the form $\{v | u \leq^k v\}$ for some $u \in A^+$. By Lemma 3.1(v), $\{v | u \leq^k v\} = L^+_{\phi^k_u}$, hence $L^+_\phi = L^+_\psi$, where ψ is a finite disjunction of sentences of the form ϕ^k_u.

(iii) and (iv) are proved similarly to Lemma 2.3(ii),(iii); one has only to keep in mind that, by the preceding paragraph, any $\phi \in E^k$ is a finite disjunction of sentences of the form ϕ^k_u. $\square$

The analog of Lemma 2.3(iv) is more intricate. In the following assertion the boldface letters have the same meaning as in Section 2.

Lemma 3.3. *(i) If* $\mathbf{u} \subseteq \mathbf{v} \subseteq^k \mathbf{U}$ *and* $\mathbf{u} \subseteq^k \mathbf{U}$ *then* $\mathbf{u} \subseteq^k \mathbf{v}$.

(ii) $\phi \in B(E^k)$ *iff* $\forall \mathbf{U} \exists \mathbf{u} \subseteq \mathbf{U} (\forall \mathbf{v} (\mathbf{u} \subseteq \mathbf{v} \subseteq^k \mathbf{U} \rightarrow \mathbf{v} \models \phi) \vee \forall \mathbf{v} (\mathbf{u} \subseteq \mathbf{v} \subseteq^k \mathbf{U} \rightarrow \mathbf{v} \models \neg\phi))$.

Proof. (i) is straightforward.

(ii) From right to left, the proof is the same as in Lemma 2.3(iv), hence we consider only one direction. Let $\phi \in B(E^k)$, then $\phi \in D^k_n$ for some n, say $n = 2$. Let $\phi_0, \phi_1 \in E^k$ satisfy $\phi \equiv \phi_0 \wedge \neg\phi_1$, and ϕ_1 implies ϕ_0. For any CLO-model $\mathbf{U}$ we have to find $\mathbf{u} \subseteq \mathbf{U}$ satisfying the disjunction from (ii). In the case $\mathbf{U} \models \neg\phi_0$, we can take any $\mathbf{u} \subseteq \mathbf{U}$ (use 3.2(iii) to show that the second disjunction in (ii) holds true). In the case $\mathbf{U} \models \phi$, represent ϕ_0 as a disjunction of the sentences ϕ_u^k; so $\mathbf{U}$ satisfies one of these ϕ_u^k. Then $\mathbf{u} \subseteq \mathbf{U}$, and from (i) it follows that the first member of the disjunction from (ii) holds true. The remaining case $\mathbf{U} \models \phi_1$ is treated similarly to the previous one; in this case the second member of the disjunction from (ii) holds true. $\square$

Repeating now the argument from Section 2, we get the following generalization of Theorem 2.2 (by Lemma 3.1.(iii), Theorem 2.2 is obtained if one takes $k = 0$).

Theorem 3.2. *For all k and n, the classes D_n^k and $B(E^k)$ are decidable.*

Now let us show that E^{k+1} contains many new sentences as compared with D_n^k.

Lemma 3.4. *If the alphabet A contains at least two letters then $E^{k+1} \not\subseteq B(E^k)$ for any k.*

Proof. Let $a, b \in A, a \neq b$. For $k = 0$, consider the sentence $\phi = Q_a(\top), \phi \in E^1, L_\phi^+ = A^*a$. The sequence $a, ab, aba, abab, \ldots$ is an infinite alternating $\leq^0$-chain for ϕ, hence $\phi \notin B(E^0)$.

For $k > 0$, consider the sentence ϕ saying that the last $k+1$ letters are a, so $L_\phi^+ = A^*a^{k+1}$. Of course, $\phi \in E^{k+1}$. Define words $u_i (i \geq 0)$ as follows: $u_0 = a^k a^k, u_{2i+1} = u_{2i}ba^k, u_{2i+2} = u_{2i+1}a^k$. It is easy to see that $\{u_i\}$ is an infinite alternating $\leq^k$-chain for ϕ. Hence, $\phi \notin B(E^k)$. $\square$

Now we shall relate the classes D_n^k and D_n^+. Let $n \geq 1, w_1, \ldots, w_n \in A^+, l_i = |w_i|$ and $w_1 \ldots w_n = a_1 \ldots a_m (a_j \in A, m = l_1 + \cdots + l_n)$. Let $\phi(w_1, \ldots, w_n)$ be a Σ_1^0-sentence of signature σ'' saying that there exist $x_1 < \cdots < x_m$ such that $x_1 = \perp, x_m = \top, Q_{a_i}(x_i)$ for $i = 1, \ldots, m$ and $S(x_i, x_{i+1})$ for $i \in \{1, \ldots, m\} \setminus \{l_1, l_1 + l_2, \ldots, l_1 + \cdots + l_{n-1}\}$.

Lemma 3.5. *(i) $u \models \phi(w_1, \ldots, w_n)$ iff $u = w_1v_1w_2v_2 \ldots w_n$ for some $v_1, \ldots, v_{n-1} \in A^*$.*

(ii) For any $\phi \in \Sigma_1^0, L_\phi^+ \neq \emptyset$, there is a disjunction ψ of sentences of the form $\phi(w_1, \ldots, w_n)$ satisfying $L_\psi^+ = L_\phi^+$.

Proof. (i) holds by definition of $\phi(w_1, \ldots, w_n)$.

(ii) Let ϕ be $\exists\bar{y}\psi(\bar{y})$, where ψ is a quantifier–free formula of signature σ''. We may assume that ψ is a conjunction of atomic formulas and of negated atomic formulas (otherwise convert ψ to the disjunctive normal form and distribute $\exists$ through $\vee$; this standard algorithm is used below as well). As observed in [Th82], we may assume that ψ is equivalent to $\perp = z_1 \leq \cdots \leq z_m = \top \wedge \psi(\bar{z})$, for some permutation $(z_1, \ldots, z_m)$ of the variables $\bar{y}$. It is clear that ϕ is further reducible to a disjunction of sentences of the form

$$\exists x_1, \ldots, x_m(\perp = x_1 < \cdots < x_m = \top \wedge \theta), \tag{1}$$

where θ is again a conjunction of atomic formulas and of negated atomic formulas.

The negated atomic formulas may be eliminated as follows. If θ contains a formula $\neg(x_i = x_i)$ then (1) is false. Formulas $\neg(x_i = x_j)$ for $i \neq j$ are true, hence they may be omitted from θ. Formulas $\neg(x_i < x_j)$ are eliminated in the same fashion. The formula $\neg Q_a(x_i)$ is replaced by disjunction of formulas $Q_b(x_i), b \in A \setminus \{a\}$ (with the subsequent distribution of $\exists$ through $\vee$). Formulas $\neg S(x_i, x_j)$ for $j \neq i+1$ are true, hence they may be omitted from θ. Finally, the formula $\neg S(x_i, x_{i+1})$ is replaced by the equivalent formula $\exists z(x_i < z < x_{i+1})$, for a new variable z, with the subsequent move of $\exists z$ to the prefix (note that the last operation reduces the number of formulas $\neg S(x_i, x_{i+1})$ in θ, hence we may use the induction).

In this way, we get a disjunction of sentences (1), where θ is a conjunction of atomic formulas. Note that the formula $\bigvee_{a \in A} Q_a(x_i)$ is true, hence we may assume that θ is of the form $Q_{a_1}(x_1) \wedge \ldots \wedge Q_{a_m}(x_m) \wedge \theta'$, where $a_i \in A$ and θ' is a conjunction of formulas $S(x_i, x_j), x_i < x_j$ and $x_i = x_j$. The two last types of formulas, as well as formulas $S(x_i, x_j)$ for $j \neq i+1$, are again eliminated in the obvious way.

Hence, we may assume that θ' is a conjunction of formulas $S(x_i, x_{i+1})$ for several $i \in \{1, \ldots, m-1\}$. Let $m_1 < \cdots < m_{n-1}$ be all numbers $i \in \{1, \ldots, m-1\}$ for which the formula $S(x_i, x_{i+1})$ is not a member of the conjunction θ'. Define the words

$$w = a_1 \ldots a_m, w_1 = w[1, m_1], w_2 = [m_1 + 1, m_2], \ldots, w_n = [m_{n-1} + 1, m].$$

Then (1) is equivalent (over finite words) to $\phi(w_1, \ldots, w_n)$ completing the proof. $\Box$

Now we can state the desired relationship.

Lemma 3.6. *(i)* $S_1^+ = \cup_k E^k$.
(ii) For any n, $D_n^+ = \cup_k D_n^k$.
(iii) $B(S_1^+) = \cup_{n,k} D_n^k$.

Proof. (ii) and (iii) easily follow from (i), hence we check only (i). By definition of E^k, $E^k \subseteq S_1^+$, hence it remains to check the inclusion $S_1^+ \subseteq \cup_k E^k$. By Lemma 3.5(ii), it suffices to show that $\phi = \phi(w_1, \dots, w_n) \in \cup_k E^k$ for all $n \geq 1$ and $w_1, \dots, w_n \in A^+$. We claim that $\phi \in E^k$, where $k = max\{|w_1|, \dots, |w_n|\}$. By Lemma 3.2(ii), it suffices to check that L_ϕ^+ is closed upwards under $\leq^k$. Assume that $u \models \phi$ and $u \leq^k v$, By Lemma 3.5(i), $u = w_1 v_1 w_2 v_2 \dots w_n$ for some $v_1, \dots, v_{n-1} \in L^*$. From the definition of $u \leq^k v$ and the choice of k, it follows that $v = w_1 y_1 w_2 y_2 \dots w_n$ for some $y_1, \dots, y_{n-1} \in A^*$. Hence, $v \models \phi$. □

Theorem 3.2 together with a result from [St85] implies

Corollary 3.1. *The class $B(S_1^+)$ is decidable.*

Proof. From [St85, Theorem 1.7] and Lemma 3.2, it follows that, given a σ'-sentence ϕ, one can effectively find k and n such that $\phi \in B(S_1^+)$ if and only if $\phi \in D_n^k$. It remains to apply Theorem 3.2. □

Corollary is equivalent to the well-known result that the class of so-called languages of dot-depth one is decidable.

Remark 3.2. Unfortunately, the results of this section are not so complete and elegant as those of Section 2. The proof of the corollary is not completely satisfactory from the viewpoint of methodology of our paper, because it uses an automata–theoretic argument (in the proof of the cited result from [St85]).

4. THE EMPTY WORD

Here we relate the hierarchies considered above to the "real" Straubing and Brzozowski hierarchies which classify subsets of A^* (rather than A^+). We state a simple relationship that aims to avoid annoying discussions (and sometimes even confusions) caused by the role of the empty word ε in this context.

The Straubing hierarchy is defined as follows (see [PP86]): $\mathcal{B}_0 = \mathcal{A}_0 = \{\emptyset, A^*\}$; let $\mathcal{B}_{n+1}$ be the closure of $\mathcal{A}_n$ under $\cap, \cup$, and the operation which relates the concatenation language XaY to languages X, Y and a letter $a \in A$; finally, let $\mathcal{A}_{n+1} = B(\mathcal{B}_{n+1})$ be the Boolean closure of $\mathcal{B}_{n+1}$. The sequence $\{\mathcal{B}_n\}$ is known as *Straubing hierarchy.*

In [PP86], a natural logical description of the introduced classes of languages was established. Namely, the classes of sentences Σ_n and Γ_n were found such that $\mathcal{B}_n = \{L_\phi | \phi \in \Sigma_n\}$ and $\mathcal{A}_n = \{L_\phi | \phi \in \Gamma_n\}$. Here L_ϕ

is defined similarly to the language L^+_ϕ of Section 2, but now the empty structure is also admitted (with a natural notion of satisfaction).

Let $\mathcal{S}_n = \{L^+_\phi | \phi \in S^+_n\}$, where S^+_n is the class from Section 2. For $\mathcal{X} \subseteq P(A^+)$, let $\mathcal{X}^\varepsilon = \{X \cup \{\varepsilon\} | X \in \mathcal{X}\}$. Then the desired relationship between the introduced classes is as follows.

Theorem 4.1. *For any $n > 0$, $\mathcal{B}_n = \mathcal{S}_n \cup \mathcal{S}^\varepsilon_n$ and $\mathcal{A}_n = B(\mathcal{S}_n) \cup B(\mathcal{S}_n)^\varepsilon$.*

Proofsketch. First note that $\mathcal{S}_n \subseteq \mathcal{B}_n$ (if $X \in \mathcal{S}_n$, then $X = L^+_\phi$ for a sentence $\phi \in S^+_n \subseteq \Sigma_n$ starting with the existential quantifier; hence $\varepsilon \not\models \phi$ and $X = L_\phi \in \mathcal{B}_n$.)

The desired equalities are checked by induction on n. We have already proven that $\mathcal{S}_1 \subseteq \mathcal{B}_1$. Note that $\{\varepsilon\} = L_\phi$, where ϕ is $\forall x(x \neq x)$, hence $\{\varepsilon\} \in \mathcal{B}_1$. But $\mathcal{B}_1$ is closed under $\cup$, so $\mathcal{S}^\varepsilon_1 \subseteq \mathcal{B}_1$ and $\mathcal{S}_1 \cup \mathcal{S}^\varepsilon_1 \subseteq \mathcal{B}_1$.

For the converse, recall that $\mathcal{B}_1$ is a closure of $\mathcal{A}_0$, hence, to prove the inclusion $\mathcal{B}_1 \subseteq \mathcal{S}_1 \cup \mathcal{S}^\varepsilon_1$, it suffices to show that the class $\mathcal{S}_1 \cup \mathcal{S}^\varepsilon_1$ contains $\mathcal{A}_0$ and is closed under $\cup, \cap$, and the operation XaY. Only the last assertion is not evident, so let us deduce $XaY \in \mathcal{S}_1 \cup \mathcal{S}^\varepsilon_1$ from $X, Y \in \mathcal{S}_1 \cup \mathcal{S}^\varepsilon_1$. By the cited result from [PP86], $X = L_\phi$ and $Y = L_\psi$ for some $\phi, \psi \in \Sigma_1$. Let θ be $\exists x(Q_a(x) \wedge \phi^{(<x)} \wedge \psi^{(>x)})$, where $\phi^{(<x)}$ and $\psi^{(>x)})$ are evident relativizations of ϕ and ψ, respectively. By definition of Σ_1 [PP86], $\theta \in S^+_1$, hence $XaY \in \mathcal{S}_1$.

The equality $\mathcal{A}_1 = B(\mathcal{S}_1) \cup B(\mathcal{S}_1)^\varepsilon$ is easy, which completes the induction basis. The argument of induction step is almost the same as for the basis. $\square$

Let $\{\mathcal{D}_{n,k}\}_k$ be the difference hierarchy over $\mathcal{S}_n$ and $\{\mathcal{D}'_{n,k}\}_k$ be the difference hierarchy over $\mathcal{B}_n$. Using Theorem 4.1 and an evident set-theoretic argument, we get

Corollary 4.1. *For all n and k, $\mathcal{D}'_{n,k} = \mathcal{D}_{n,k} \cup \mathcal{D}^\varepsilon_{n,k}$.*

There is a similar relationship between the Brzozowski hierarchy and the corresponding classes from Section 3.

5. CONCLUSION

We see that some problems of the automata theory not only may be formulated in a logical form, but they can even be solved by logical means. It is natural to ask a general logical question generalizing problems considered in Sections 2 and 3. For a given theory T, let S_n be the set of sentences equivalent, in the theory T, to a Σ^0_n-sentence. Let S^+_n be defined similarly but using the equivalence in finite structures. One can also define the classes

$D_{n,k}(D^+_{n,k})$ of the difference hierarchy over S_n (respectively, over S^+_n), and even the classes of the fine hierarchy over $\{S_n\}$ (see [Se91, Se95]).

The general question is to determine in what cases the introduced classes of sentences are decidable. The problems considered in Sections 2 and 3 are obtained when one considers the theories CLO and CLO' in place of T.

The question is quite traditional for mathematical logic, hence one could hope to find some relevant information in the corresponding literature. Indeed, in [Ma71] we find (with the reference to source papers) the following result: if T is undecidable, then so are S_n for all $n > 0$. But what about the more interesting case of a decidable theory T (which is the case for CLO and CLO')? It seems that, strangely enough, there is almost nothing known about this natural problem. From the results of [Se91a, Se92] (which rely upon the Tarski elementary classification of Boolean algebras), one can easily deduce the following result.

Theorem 5.1. *Modulo theory T of Boolean algebras, all classes $D_{n,k}$ (and even all classes of the fine hierarchy) are decidable.*

Proof. In [Se91a, Se92] we have described an effective sequence of sentences $\phi_0, \phi_1, \ldots$ such that any sentence ϕ is equivalent (modulo theory of Boolean algebras) to exactly one of ϕ_i, and the position of any ϕ_i in the hierarchy $\{D_{n,k}\}$ is completely determined. This evidently implies the desired algorithm. $\square$

It seems interesting to consider analogs of Theorem 5.1 for other popular decidable theories, say, for abelian groups.

We hope that the methods developed in this paper may be used in some other similar situations, say, for the case of tree languages.

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ЛОГИЧЕСКИЙ ПОДХОД К РАЗРЕШИМОСТИ ИЕРАРХИЙ РЕГУЛЯРНЫХ БЕЗЗВЕЗДНЫХ ЯЗЫКОВ

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